Chapter 1 Introduction to Standard Colorimetry

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Abstract

Colorimetry is the branch of color science concerned with the numerical specification of color. The main focus of colorimetry has been the development of methods for predicting perceptual color matches on the basis of radiometric (physical) measurements.
In this chapter, starting from the Grassman laws and the first color-matching experiments, we will introduce the LMS fundamental reference, the RGB instrumental reference and the XYZ CIE 1931 standard system.
After discussing the non-uniformity of the well-known CIE chromaticity diagram, we will present the two CIE psychometric color systems, termed CIELAB and CIELUV, respectively.
Finally, we will explore the development of the most recent color difference formulae.
For the scientists keen in programming, we will also present some of the basic colorimetric numerical calculations.

Keywords:

Psychophysical Colorimetry, Psychometric colorimetry, CIE Systems, Color difference formulae, C++ algorithms.

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1. Introduction

Colorimetry is constituted by two main parts, psychophysical and psychometric colorimetry, and both parts have their origin in visual physiology. A general knowledge of the human visual system is needed.
In colorimetry, the eye can be considered as a spherical optical chamber.
Light is measured outside the eye and the visible range of wavelength is $360 \div 780$ nm. The light entering the eye is named color stimulus and is physically represented by a spectral radiance $L_e(\lambda)[\mathrm{W/(sr\, m^2 nm)}]$ (Wyszecki, G. and Stiles, W. S. (2000).
Light enters the eye, crosses some different media and is focused on the back of the eye, that is covered by a membrane sensitive to light, named retina. The spectral power distribution of the light is altered by the lens of the eye, that is strongly absorbent in the short wavelength region below 450 nm.
The retina is a non-uniform tissue of many layers of different kinds of cells with proper roles in the visual process. The external layer of cells is composed of photoreceptors, thus the light can be absorbed by the photoreceptors only after travelling across almost all the retina. In particular, the central part of the retina, named macula lutea, contains an inert pigment that modifies the spectral power distribution of the light. Thus, the light that crosses the photoreceptors is altered differently in the macular region with respect to the non-macular region. Such a difference leads to define two different colorimetric systems, one for the macular region, and the other for the region external to the macula. The visual field related to macular vision has an angular section lower than 4°, generally 2°, while extra macula vision an angular section of 10°. Generally, this angle is called visual angle.
The photoreceptors are subdivided into rods and cones, as their shapes suggest. The rods are dedicated to crepuscular vision, named scotopic vision, while the cones are dedicated to color vision, named photopic vision.
We are interested only in color vision, thus we consider the cones, that are of three different kinds characterized by different photopigments. The absorption of light by the cones is the first step in visual processing and is named cone activation. The cones with maximum absorption of light in the short, medium and long wavelength region are named cones S, cones M and cones L, respectively.

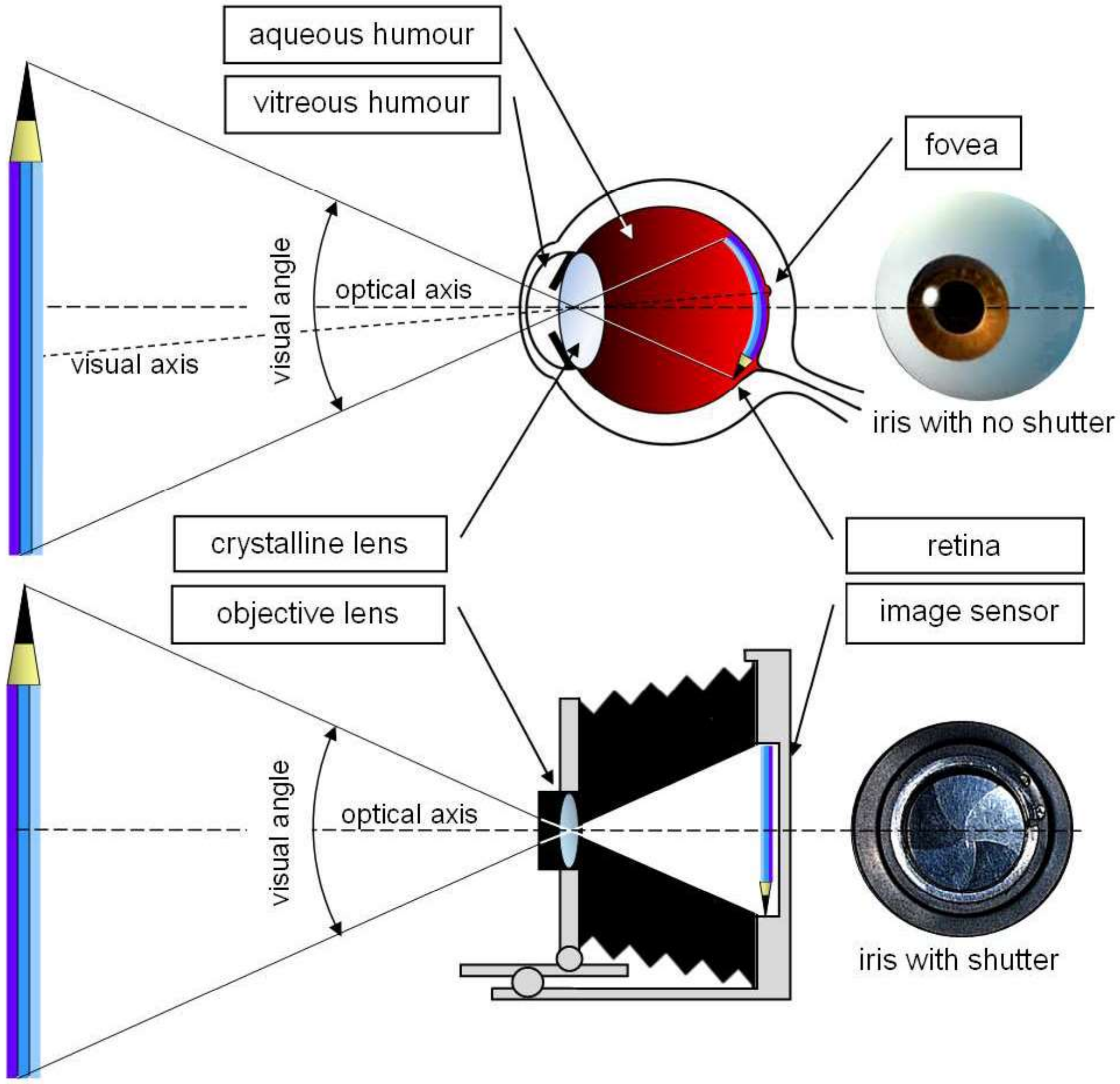

Figure 1. Comparison of the eye with a camera. Figure reproduced with permission from Oleari C. (2016).

The light absorbed by cones produces cone activation. Cone activation is based on Rushton's univariance principle, that says that the visual effect of radiation depends only on the number of absorbed photons and is independent of the photon energy, i.e., a photon, once absorbed by the pigment of a cone, triggers a photochemical process, that is independent of the photon energy. Photons with different energy have their own probability to be absorbed and consequently produce different visual effects. Cone activation is the first step in the color vision phenomenon and is represented by the numbers of photons absorbed by the three kinds of cones in the unit of time. In any defined visual situation, the correspondence between cone activations and color sensations is one to one. Thus, cone activation can be used to specify color.

First, the other layers of cells of the retina make a comparison between the signals of the cones generated locally in the retina. The result of this comparison depends on the signals generated in the proximal field and the color sensation of a point depends on the global visual situation. This signal processing, that constitutes the second step in the color vision phenomenon, is not yet completely understood and is in part linear and in part non-linear. These stages of the color-vision phenomenon correspond to the zones of Mueller's zone theory.
The picture here given of the retina is sufficient to define different colorimetric systems as functions of the retinal region and as functions of the stage of visual processing.
This simple system of three photoreceptors is very powerful and gives a wavelength discrimination between 1 and 3 nm in a wide part of the visible range.

2. Psychophysical colorimetry

Let us consider the cone activations, the first stage of the visual phenomenon in the macular region.
The cone activations are represented by a set of three numbers that satisfy linear addition (Grassmann's laws) and are well represented by points (vectors) in a three-dimensional linear space, known as tristimulus space. In this case the reference frame of the tristimulus space is named fundamental reference frame (Wyszecki, G. and Stiles, W. S. (2000), Grassmann H. (1853), Grassmann H. (1853), Wyszecky G. and Judd D.B. (1975)).
Many reference frames are possible in the tristimulus space, of which let us recall:

- The LMS fundamental reference.
- The RGB instrumental reference.
- The XYZ CIE 1931 standard system.

For didactic reasons, we introduce, first, the tristimulus space with the fundamental reference frame and then, departing from this, the RGB and the more used XYZ CIE 1931 system.

2.1 Tristimulus space and the fundamental reference frame

The cone activations produced by a color stimulus with spectral radiance $L_e(\lambda)$ are proportional to three numbers *(L, M, S)*, named tristimulus values, and representing the activations of the cones *L, M* and *S*, respectively:

$$L = \int_{380}^{780} L_e(\lambda) \cdot \bar{l}(\lambda) d\lambda, M = \int_{380}^{780} L_e(\lambda) \cdot \bar{m}(\lambda) d\lambda, S = \int_{380}^{780} L_e(\lambda) \cdot \bar{s}(\lambda) d\lambda,$$

where the functions $\bar{l}(\lambda), \bar{m}(\lambda), \bar{s}(\lambda)$ (Figure 2), named color-matching functions (CMF), are the spectral sensitivities of the cones L, M and S, respectively, and take into account also the light absorption of the lens of the eye and of the macula lutea. The color-matching functions are normalized to give $L = M = S = 1$ for the equal-energy radiance $L_e(\lambda) = 1$, thus represent the monochromatic components with unitary radiance of the equal-energy stimulus ($\bar{l}(\lambda)$, $\bar{m}(\lambda), \bar{s}(\lambda)$) in the tristimulus space.

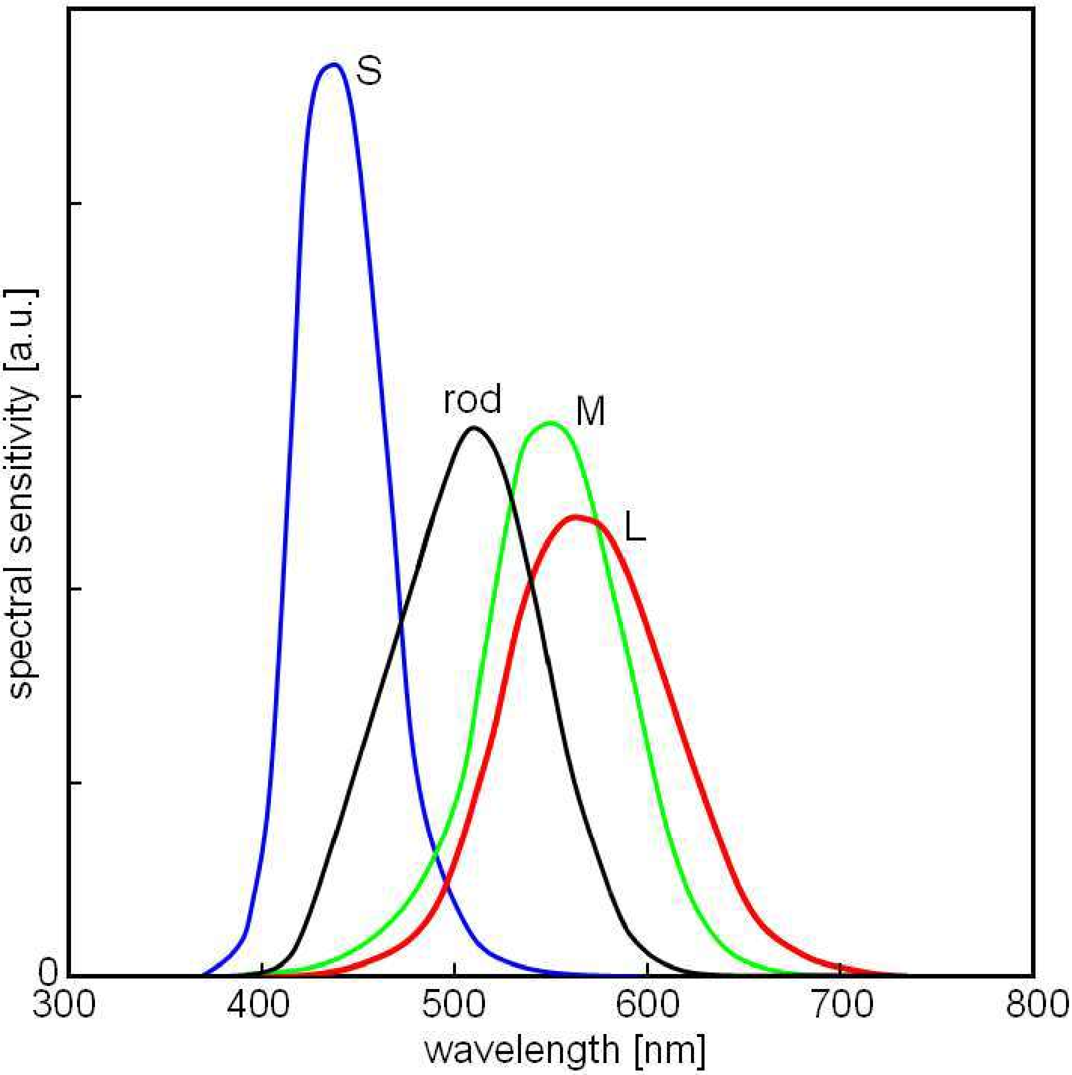

Figure 2. Color-matching functions in the fundamental reference frame.

The mathematical properties of the tristimulus space are defined for the first time by Grassmann's laws, although the original idea is Newton's centre of gravity rule.
The sum of color stimuli is represented in the tristimulus space by the sum of the corresponding vectors (Figure 3). The definition of the (L, M, S) vectors claims that the correspondence between color stimuli and tristimulus vectors is many to one, i.e., different radiances can produce equal cone activations and equal color sensation. This phenomenon is named metamerism and the color stimuli producing the same color sensation are named metamers. The CIE definition of metamerism regards the "metameric colour stimuli: spectrally different color stimuli that have the same tristimulus values in a specified colorimetric system" (Publication CIE S 017/E:2011 (2011)).
The length of the vectors is related to the intensity of the color stimuli and the direction is related to chromatic sensation. Since the vector directions are in a one-to-one correspondence with the intersection points between the tristimulus vectors and a plane, these points constitute a diagram, termed chromaticity diagram because it represents the chromaticity (Figure 3). The chromaticities of the monochromatic lights are points of a line named spectrum locus and the segment between short wavelength and long wavelength regions regards the purple hues. The practical role of the chromaticity diagram is very important.
The coordinates system on the plane of the chromaticity diagram in order to define the chromaticity of the color stimuli is as follows:

$$l = \frac{L}{L + M + S},$$
$$m = \frac{M}{L + M + S},$$
$$s = \frac{S}{L + M + S} = 1 - l - m.$$

These coordinates are called chromaticity coordinates.

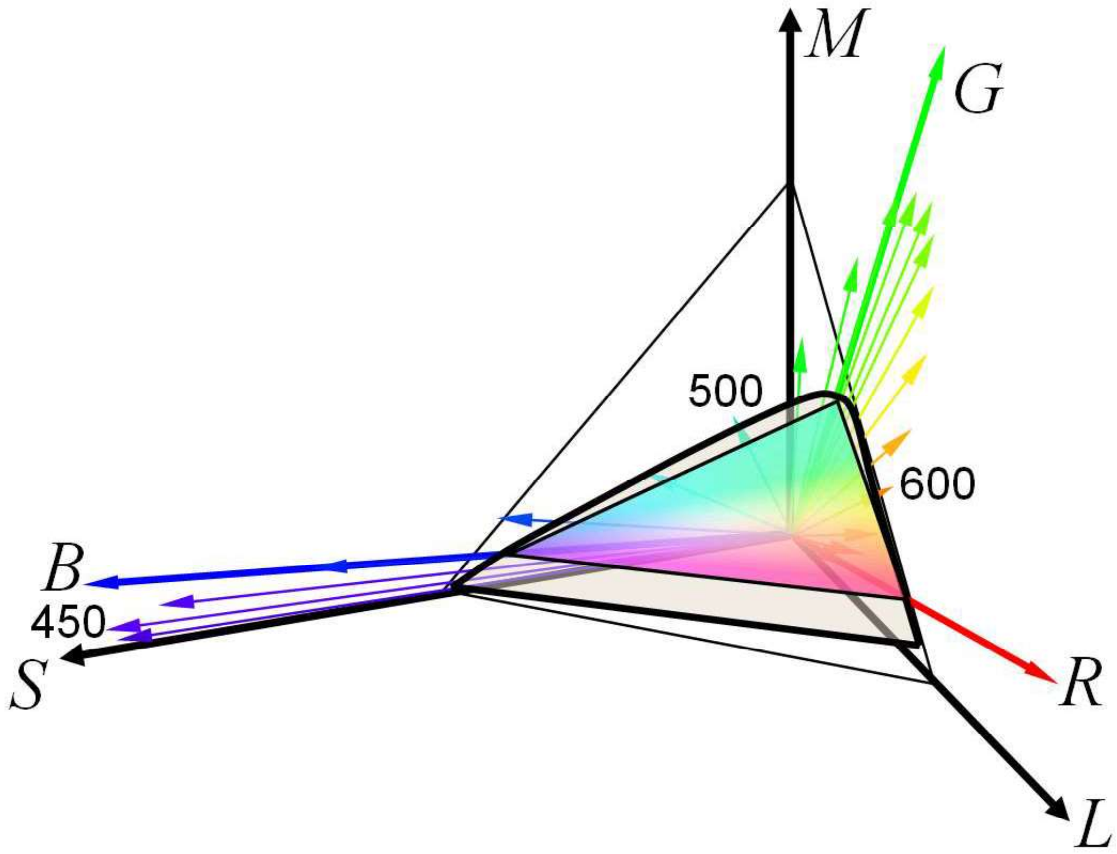

Figure 3. Fundamental reference frame in the tristimulus space. On the plane $L + M + S = 1$ is defined the chromaticity diagram (grey area). Figure reproduced with permission from Oleari C. (2016).

2.2 Tristimulus space and instrumental RGB reference frame

The color-matching functions are not measured by probes introduced in the cones, but are measured indirectly by a technique known as color-matching. This technique needs the choice of three lights (generally monochromatic) as reference lights (often called primary lights, although contradicting CIE [21]), generally one red, one green and one blue, whose corresponding tristimulus vectors constitute the reference frame in the tristimulus space.

In color matching, in correspondence to any monochromatic light of wavelength λ and of unit radiance, two lights are shown to the observer: the first light is obtained by mixing the two reference lights, for which the monochromatic light is in between, and the second light is obtained by mixing the monochromatic light with the third reference light. The observer modifies the radiances L_R, L_G and L_B of the three reference lights until a match is obtained. The values of the radiances $\overline{r}(\lambda) = L_R$, $\overline{g}(\lambda) = L_G$ and $\overline{b}(\lambda) = L_B$ represent the CMFs at the wavelength λ. Generally, after measuring these

CMFs, they are normalized in order to obtain equal tristimulus values for the equal-energy stimulus, i.e., the stimulus of a spectral radiance constant in wavelength. The tristimulus values in the RGB reference frame are defined by the integrals:

$$R = \int_{380}^{780} L_e(\lambda) \cdot \bar{r}(\lambda) d\lambda,$$

$$G = \int_{380}^{780} L_e(\lambda) \cdot \bar{g}(\lambda) d\lambda,$$

$$B = \int_{380}^{780} L_e(\lambda) \cdot \bar{b}(\lambda) d\lambda,$$

and the chromaticity coordinates are represented by barycentric coordinates (r, g, b):

$$r = \frac{R}{R+G+B},$$

$$g = \frac{G}{R+G+B},$$

$$b = \frac{B}{R+G+B} = 1 - r - g.$$

A linear transformation exists between the tristimulus values (L, M, S) and *(R, G, B)*, as well as between the color-matching functions ($\bar{l}(\lambda),\ \bar{m}(\lambda), \bar{s}(\lambda)$) and ($\bar{r}(\lambda),\ \bar{g}(\lambda), \bar{b}(\lambda)$). This linear transformation can be realized by the knowledge of the LMS reference axes in the RGB reference frame, which is obtained from the dichromat color matching. The dichromats are observers with only two kinds of cones.
The RGB reference frame here considered regards the CIE 1931 standard observer. A standard observer is the hypothetical individual whose color-matching behavior is represented by the set of standard CMFs.
Any trichromatic device (monitor, scanner, video camera, ..., TV system like NTSC, PAL, SECAM, HDTV, ...) has its own RGB reference frame and the passage between different RGB reference frames is made by linear transformations. Confusing different RGB reference frames is a mistake. Moreover, an RGB system exists, whose (R, G, B) components are obtained as powers of tristimulus values, destroying the original linearity of the space (e.g., sRGB used for images on computers and the web).

2.3 Tristimulus space and XYZ reference frame of the CIE 1931 system

The CIE 1931 colorimetric system was realized by embedding photometry in colorimetry. The photometry relates the sensation of luminosity of a light defining the luminance, the quantity that would represent the brightness associated the radiance $L_{e,\lambda}$:

$$L_v = K_m \int_{380}^{780} L_e(\lambda) \cdot V(\lambda) d\lambda,$$

with $V(\lambda)$ = relative photopic luminous efficiency function that defines the standard photometric observer CIE 1924 and $K_m = 683$ lumen / watt.
Abney's law states that the luminance of a color stimulus obtained as a sum of many stimuli is equal to the sum of the corresponding luminances. This law, although a weak law, induced scientists to represent luminance as a linear weighted sum of the tristimulus values:

$$L_v = L_L + L_M M + L_S S = L_R R + L_G G + L_B B.$$

The coefficients L_L, L_M, L_S, L_R, L_G and L_B are called Exner coefficients and equation Schrödinger's "Helligkeit" equation.
As a consequence, to search for a reference frame XYZ such that the Y component of the stimulus is proportional to the luminance L_v, i.e., $L_v = K_m = Y$ cd/m^2 and $\bar{y}(\lambda) = V(\lambda)$, where $(\bar{x}(\lambda),\ \bar{y}(\lambda),\ \bar{z}(\lambda))$ are the color-matching functions in the XYZ system, the following constraints are imposed:

- The *X*, *Y* and *Z* axes are mutually orthogonal.
- The *X* and *Z* axes belong to the zero luminance plane *Y* = *0*.
- The tristimulus vectors with physical meaning have all positive components.
- The planes *X* = *0* and *Z* = *0* are tangent to the spectrum locus in the short and in the long wavelength region, respectively.

This reference frame is possible and is obtained from the RGB one by a linear transformation. This reference frame is the XYZ of the CIE 1931 standard colorimetric observer (Figure 4). The tristimulus values are:

$$X = \int_{380}^{780} L_e(\lambda) \cdot \bar{x}(\lambda) d\lambda,$$
$$Y = \int_{380}^{780} L_e(\lambda) \cdot \bar{y}(\lambda) d\lambda = \int_{380}^{780} L_e(\lambda) \cdot V(\lambda) d\lambda = \frac{L_v}{K_m},$$
$$Z = \int_{380}^{780} L_e(\lambda) \cdot \bar{z}(\lambda) d\lambda,$$

and the chromaticity:

$$x = \frac{X}{X + Y + Z},$$
$$y = \frac{Y}{X + Y + Z},$$
$$z = \frac{Z}{X + Y + Z} = 1 - x - y.$$

The usual chromaticity diagram is obtained from the diagram on the plane $X = Y = Z = 1$ by a projection from infinity on the plane $Z = 0$ (Figure 5).

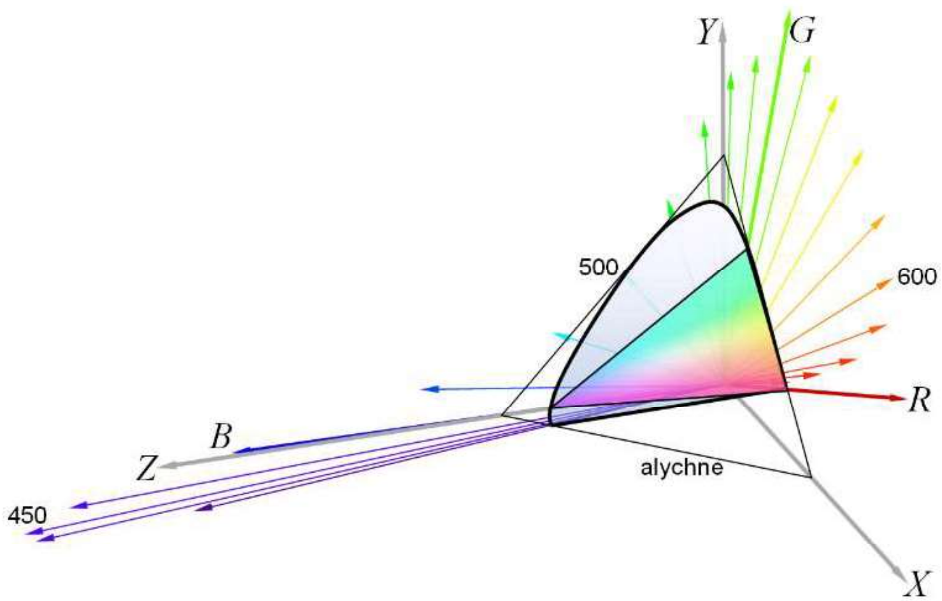

Figure 4. XYZ reference frame in the tristimulus space in which the vectors X, Y and Z constitute a set of three orthogonal vectors. The chromaticity diagram is on the plane $X + Y + Z = 1$. Figure reproduced with permission from Oleari C. (2016).

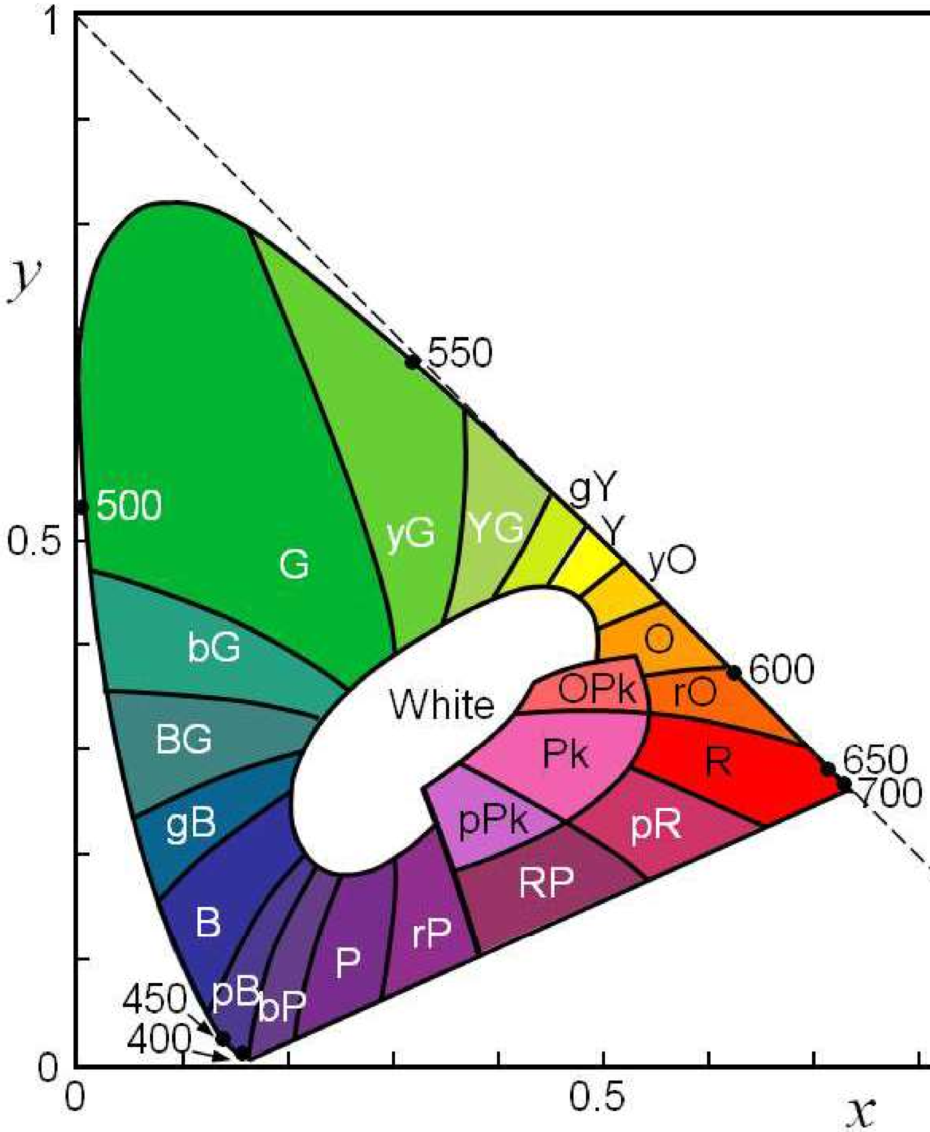

Figure 5. Chromaticity diagram CIE 1931 (x, y). Figure reproduced with permission from Oleari C. (2016).

Many other chromaticity diagrams have been proposed over the time, mainly with the intent to obtain uniform perceived scales. CIE proposed as a standard the chromaticity diagram (u', v'), which is at the basis of the psychometric system CIELUV, proposed in 1976 (see Chapter 3.x).

The CIE 1931 standard colorimetric observer is affected by a systematic error in the short wavelength region, revealed by D. B. Judd in 1951. This error has been corrected by Judd and refined by Vos (Vos J. J. (1978)) and today the corrected observer is used by physiologists. Anyway, since the correction is small, the original CIE 1931 observer is still used in industrial colorimetry today.

2.4 Supplementary Standard Observer CIE 1964

Analogous treatment can be made for extra-macula vision (visual field of 10°) and this gives the CIE 1964 system. In this case the tristimulus values are denoted by the foot-index "10", that regards the size of the visual field, i.e., by (X_{10}, Y_{10}, Z_{10}), and the color-matching functions by $(\bar{x}_{10}(\lambda), \bar{y}_{10}(\lambda), \bar{z}_{10}(\lambda))$ (Figure 6).

The shift of the chromaticities of the monochromatic radiations is remarkable, despite the apparent equality of the two diagrams (Figure 7). The luminance for the CIE 1964 observer is computed by assuming the luminous efficiency function defined by:

$$V_{10}(\lambda) = \bar{y}_{10}(\lambda).$$

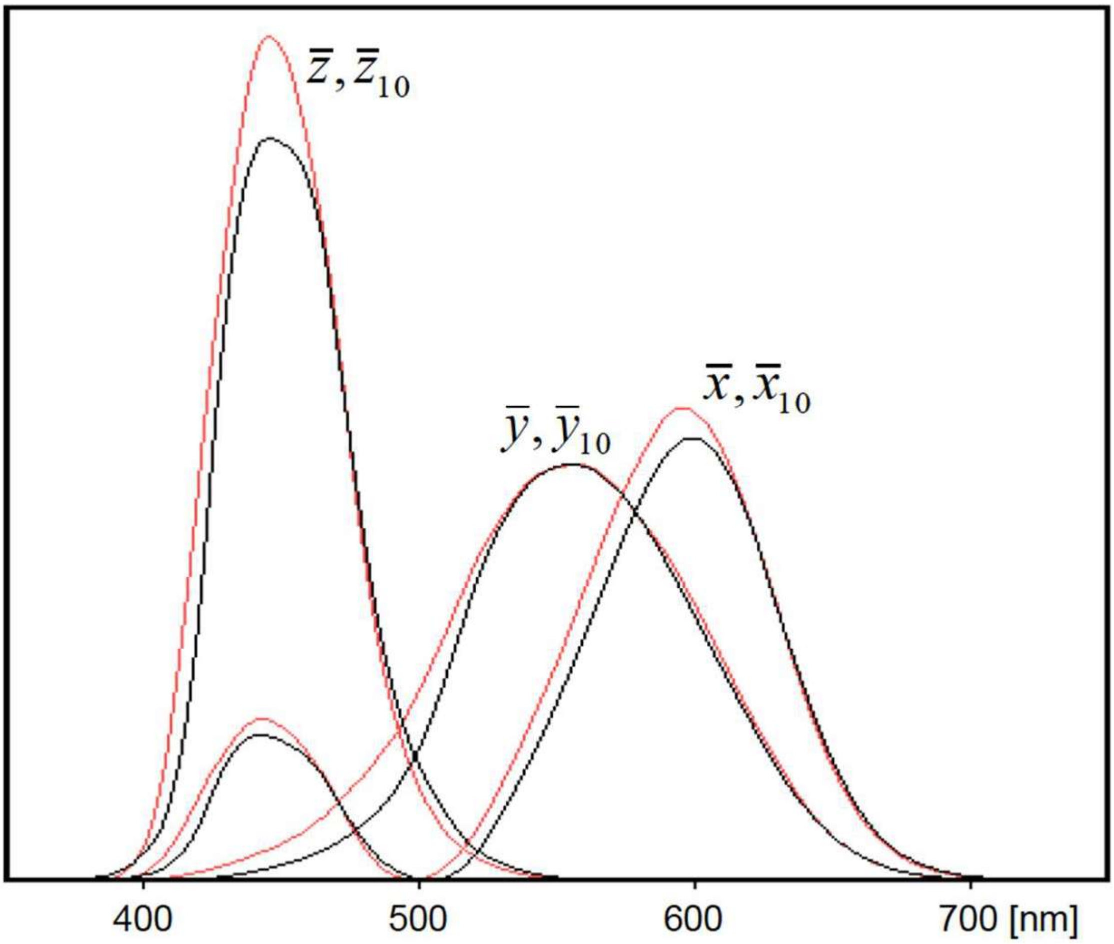

Figure 6. CIE 1931 (black line) and CIE 1964 (red line) color-matching functions.

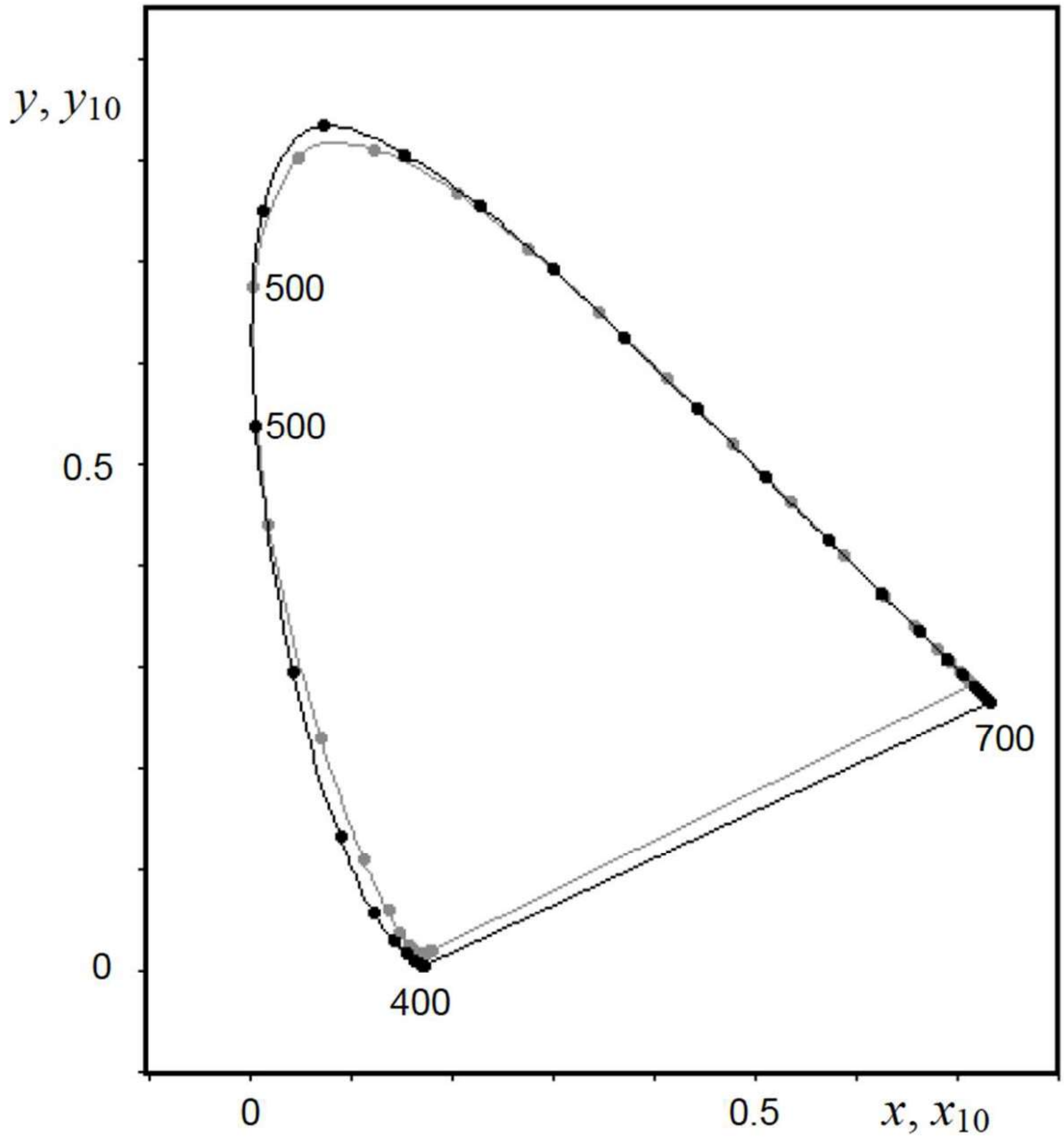

Figure 7. CIE 1931(•) and CIE 1964(•) chromaticity diagrams.

2.5 Vos Observer and MacLeod-Boynton diagram

VOS's observer has been obtained from the standard colorimetric observer CIE 1931 by correcting a systematic error present in the short wavelength region, shown by D. B. Judd in 1951 (P. K. Kaiser and R. M. Boynton (1996)). The coordinates and color-matching functions (Figure 8) are denoted as follows:

$$(X', Y', Z'), (Y', x', y'), (\overline{x}'(\lambda), \overline{y}'(\lambda), \overline{z}'(\lambda)).$$

This observer is mainly used by physiologists and is generally represented in the fundamental reference frame *(L, M, S)*.

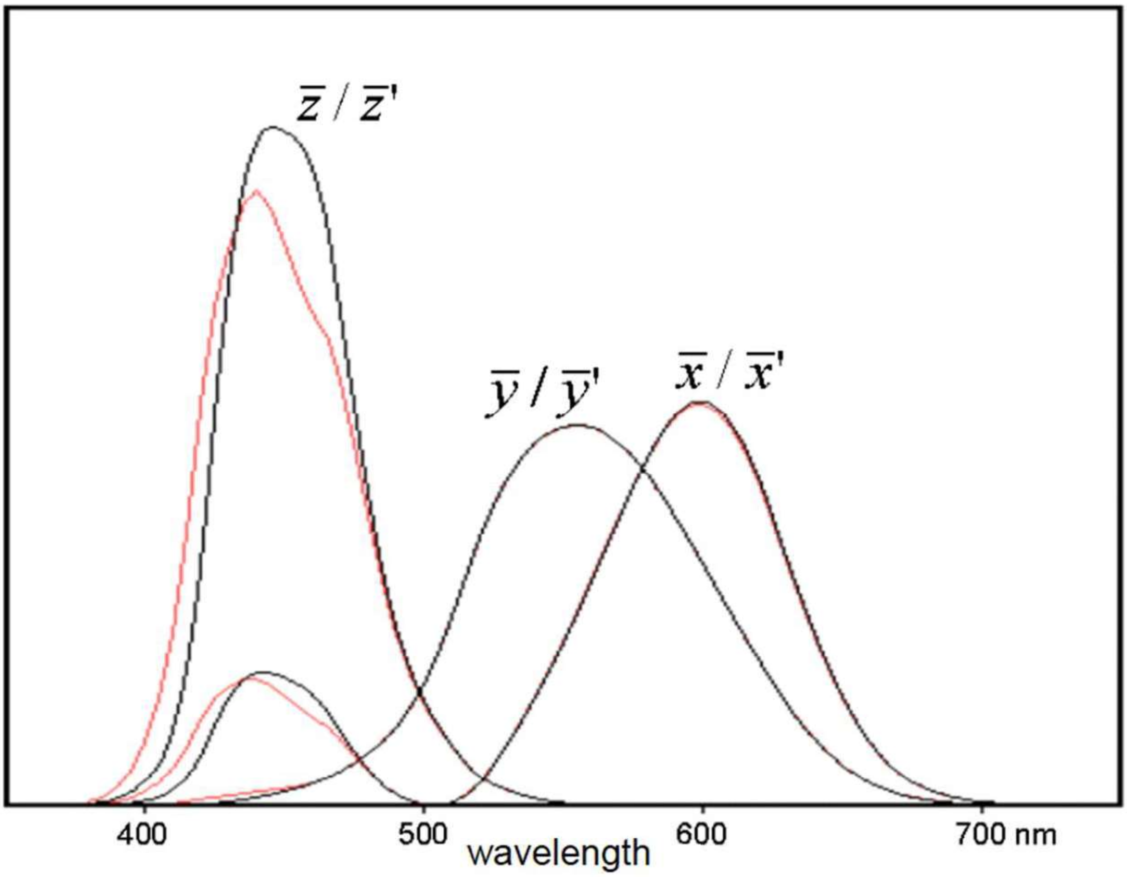

Figure 8. Comparison between VOS's (red line) and CIE 1931 (black line) color-matching functions. The discrepancy between these two sets of color-matching functions is in the short wavelength region.

The chromaticity coordinates (*l, m, s)*, introduced by MacLeod-Boynton, are defined by:

$$l = \frac{L}{L+M},$$
$$m = \frac{M}{L+M},$$
$$s = \frac{S}{L+M},$$

and satisfy the equation $l + m = 1$. The plane of this chromaticity diagram $L + M = 1$ has constant luminance, according to the hypothesis that the cone S has no contribution to the luminance, while L and M cones have equal contributions. This diagram is named the equiluminant chromaticity diagram of the cone excitations.
[N.B. these tristimulus values must not be confused with the homonymous ones defined in the previous fundamental reference frame (Section 2.1)].

2.6 Stockman and Sharpe fundamentals

In 2006 CIE accepted the CMFs proposed by Stockman and Sharpe (Stockman A. and Sharpe L. T. (2000)) as fundamentals and physiologists are now using these new observers. The XYZ reference frame is defined also for these observers (the index "F" means "fundamental", and "10" denotes the visual field of 10°):

$$\begin{pmatrix} \bar{x}_F(\lambda) \\ \bar{y}_F(\lambda) \\ \bar{z}_F(\lambda) \end{pmatrix} = \begin{pmatrix} 1.94735469 & -1.41445123 & 0.36476327 \\ 0.68990272 & 0.34832189 & 0 \\ 0 & 0 & 1.93485343 \end{pmatrix} \begin{pmatrix} \bar{l}(\lambda) \\ \bar{m}(\lambda) \\ \bar{s}(\lambda) \end{pmatrix}$$

for 2° visual field, and:

$$\begin{pmatrix} \bar{x}_{F1}\ (\lambda) \\ \bar{y}_{F10}(\lambda) \\ \bar{z}_{F10}(\lambda) \end{pmatrix} = \begin{pmatrix} 1.93986443 & -1.34664359 & 0.36476327 \\ 0.69283932 & 0.34967567 & 0 \\ 0 & 0 & 2.14687945 \end{pmatrix} \begin{pmatrix} \bar{l}_{10}(\lambda) \\ \bar{m}_{10}(\lambda) \\ \bar{s}_{10}(\lambda) \end{pmatrix}$$

for 10° visual field.

3. Color specification

Generally, an object's color is due to optical dishomogeneities of the bodies that produce absorption, diffusion, refraction and diffraction of light. The color specification of non-self-luminous objects depends on the spectral reflection or transmission of light. In the vast majority of cases, we are dealing with nearly diffuse reflection or nearly regular transmission. That is the reason that reflected flux is generally compared to the flux reflected by a perfect reflecting diffuser and that transmitted flux is generally compared to open aperture.

The physical quantity representing the phenomenon of reflection useful for the color specification is the spectral reflectance factor, as recommended by CIE Reflectance factor is the ratio of the flux reflected by a specimen to the flux reflected by a perfect reflecting diffuser under the same geometric and spectral conditions of irradiation.

The perfect reflecting diffuser is an ideal reflecting surface that is non-absorbing and non-transmitting, but it is an isotropic diffuser such that the radiance is the same in all directions and is independent of the irradiation geometry and of the wavelength. The perfect diffuser is termed Lambertian. In practice, since the perfect reflecting diffuser does not exist, a real white diffuser in used (reference standard, or a copy named working standard),

whose reflectance factor is certified by a metrological reference laboratory under equal geometries.
In this case the reflectance factor is the ratio of the flux reflected by a specimen to the flux reflected by the reference standard under the same geometric and spectral conditions, multiplied by the certified spectral reflectance factor (Publication CIE 15:2004 (2004)).

3.1 Light sources and illuminants

The color depends also on the illuminating light. An important distinction has to be made between light source and illuminant. CIE defines precise terms.

- Source: an object that produces light or other radiant flux.
- Illuminant: radiation with a relative spectral power distribution defined over the wavelength range that influences object color perception.

CIE has standardized illuminants, i.e., files representing the relative spectral power distributions $S(\lambda)$ of light sources (Figure 9). There are several important CIE standard illuminants.

- Illuminant A: associated with the tungsten lamp with the radiant exitance of the black body at the temperature of approx. 2856 K.
- Daylight type illuminants: associated with conventional daylights denoted by D50, D55, D65, D75 … at a temperature of 5000 K, 5500 K, 6500 K, 7500 K …, respectively.
- Fluorescent lamp illuminants: e.g., F2 coolwhite, F7 daylight fluorescent and F11 white fluorescent.

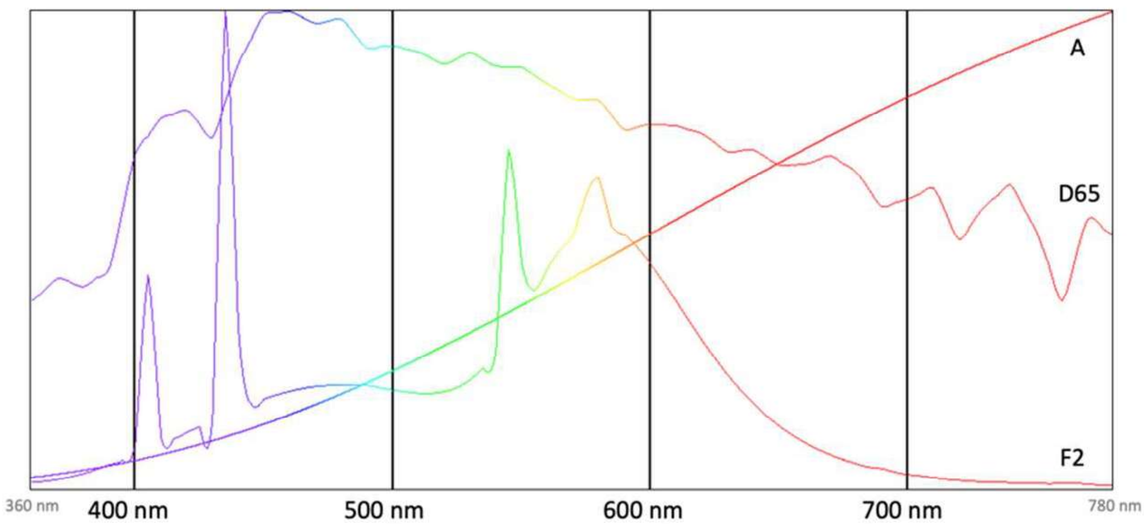

Figure 9. Spectral power distribution of illuminants A, D65 and F2 respectively. Figure created with "Colorimetric eXercise" from Oleari C. (2016).

Equi-energy spectrum or equal-energy spectrum is the spectrum of the radiation whose spectral power distribution is a constant function of the wavelength throughout the visible region. In colorimetry and photometry, the radiation with equi-energy spectrum is the equi-energy or equal-energy illuminant denoted by the symbol E.

Primary light sources and illuminants are represented by their relative spectral power distribution $S(\lambda)$, that is proportional to radiance entering the eye. The colorimetric specification is given by:

$$X = K\sum_{\lambda=360,\Delta\lambda=1}^{830} S(\lambda)\bar{x}(\lambda)\Delta\lambda\,,$$
$$Y = K\sum_{\lambda=380,\Delta\lambda=1}^{780} S(\lambda)\bar{y}(\lambda)\Delta\lambda\,,$$
$$Z = K\sum_{\lambda=360,\Delta\lambda=1}^{830} S(\lambda)\bar{z}(\lambda)\Delta\lambda\,,$$

with:

$$K = \frac{100}{\sum_{\lambda=380,\Delta\lambda=1}^{780} S(\lambda)\bar{y}(\lambda)\Delta\lambda}\,,$$

and Y is the percentage luminance factor.

Equations regard the CIE 1931 observer, but the generalization to any observer is made by changing the color-matching functions. The summations obtained by steps of $\Delta\lambda = 5$ nm are considered practical approximations.

3.2 Non-self-luminous objects

Illuminated surfaces are represented by the spectral reflectance factor $R(\lambda)$ and the radiance entering the eye is proportional to $S(\lambda)\,R(\lambda)$. The colorimetric specification is given by:

$$X = K\sum_{\lambda=36\ \ ,\Delta\lambda=1}^{830} S(\lambda)R(\lambda)\bar{x}(\lambda)\Delta\lambda\,,$$
$$Y = K\sum_{\lambda=38\ \ ,\Delta\lambda=1}^{780} S(\lambda)R(\lambda)\bar{y}(\lambda)\Delta\lambda\,,$$
$$Z = K\sum_{\lambda=360,\Delta\lambda=1}^{830} S(\lambda)R(\lambda)\bar{z}(\lambda)\Delta\lambda\,,$$

with:

$$K = \frac{100}{\sum_{\lambda=380,\Delta\lambda=1}^{780} S(\lambda)\bar{y}(\lambda)\Delta\lambda}\,,$$

and Y is the percentage luminance factor.

Equations regard the CIE 1931 observer, but the generalization to any observer is made by changing the color-matching functions. The summations obtained by steps of $\Delta\lambda = 5$ nm are considered practical approximations.

4. CIE psychometric colorimetry

In 1976, CIE proposed two psychometric color systems, termed CIELAB and CIELUV (MacAdam D. L. (1985), Publication CIE 15:2004 (2004)), respectively, with two main intents:

1. To give a practical and intuitive color specification.
2. To give a color specification with uniform perceived scales.

CIELAB and CIELUV can be considered as multi-stage color-vision models, obtained from the psychophysical color specification (X, Y, Z) by linear and non-linear transformations and introducing the dependence to an illuminant (X_n, Y_n, Z_n), to which the observer is supposed adapted. In most cases, the specified white object color stimulus should be light reflected from a perfect reflecting diffuser illuminated by the same light source as the test object. In this case, X_n, Y_n, Z_n are the tristimulus values of the light source with Y_n equal to 100.
Here are given the specifications related to the standard colorimetric observer CIE 1931. Equal specifications are given for the standard colorimetric observer CIE 1964 and are distinguished by the foot-index 10.

4.1 CIE 1976 L*a*b*color space or CIELAB color space

CIELAB color space is a three-dimensional color space with approximately uniform scales, spanned by the rectangular coordinates (L^*, a^*, b^*), where:

- L^* is the CIE 1976 psychometric lightness.
- a^* represents approximately the red-green opponency.
- b^* represents approximately the yellow-blue opponency.

These quantities are defined by the equations:

$$L^* = 116\left[f\left(\frac{Y}{Y_n}\right)\right] - 16,$$

$$a^* = 500\left[f\left(\frac{X}{X_n}\right)^{\frac{1}{3}} - f\left(\frac{Y}{Y_n}\right)^{\frac{1}{3}}\right],$$

$$b^* = 200\left[f\left(\frac{X}{X_n}\right)^{\frac{1}{3}} - f\left(\frac{Y}{Y_n}\right)^{\frac{1}{3}}\right],$$

with:

$$f\left(\frac{X}{X_n}\right) = \begin{cases} \left(\frac{X}{X_n}\right)^{\frac{1}{3}} \text{ for } \left(\frac{X}{X_n}\right) > \left(\frac{24}{116}\right)^3 \\ \frac{841}{108}\left(\frac{X}{X_n}\right)^{\frac{1}{3}} + \frac{16}{116} \text{ otherwise} \end{cases},$$

$$f\left(\frac{Y}{Y_n}\right) = \begin{cases} \left(\frac{Y}{Y_n}\right)^{\frac{1}{3}} \text{ for } \left(\frac{Y}{Y_n}\right) > \left(\frac{24}{116}\right)^3 \\ \frac{841}{108}\left(\frac{Y}{Y_n}\right)^{\frac{1}{3}} + \frac{16}{116} \text{ otherwise} \end{cases},$$

$$f\left(\frac{Z}{Z_n}\right) = \begin{cases} \left(\frac{Z}{Z_n}\right)^{\frac{1}{3}} \text{ for } \left(\frac{Z}{Z_n}\right) > \left(\frac{24}{116}\right)^3 \\ \frac{841}{108}\left(\frac{Z}{Z_n}\right)^{\frac{1}{3}} + \frac{16}{116} \text{ otherwise} \end{cases},$$

where:

- (X, Y, Z) is the psychophysical specifications of the color stimulus considered.
- (X_n, Y_n, Z_n) is the psychophysical specifications of the achromatic stimulus of the chosen illuminant.

In a cylindrical coordinate system in the same space, the coordinates are (Figure 10):

- Psychometric lightness L^*.
- Psychometric chroma $C_{ab}^* = \sqrt{a^{*2} + b^{*2}}$.
- Psychometric hue angle $h_{ab} = \arctan\left(\frac{b^*}{a^*}\right)$.

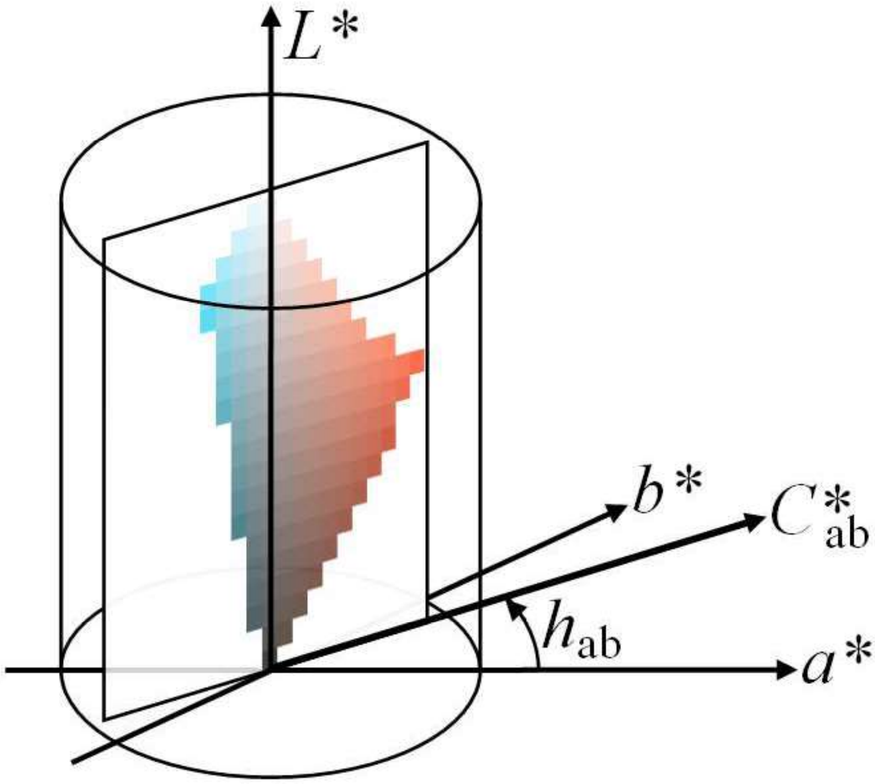

Figure 10. Perspective view of cylindrical coordinates in CIELAB space. Figure reproduced with permission from Oleari C. (2016).

4.2 CIE 1976 L*u*v*color space or CIELUV color space

CIELUV color space is a three-dimensional color space with approximately uniform scales, spanned by the rectangular coordinates (L^*, u^*, v^*), quantities defined by the equations:

$$L^* = 116\left[f\left(\frac{Y}{Y_n}\right)\right] - 16,$$
$$u^* = 13\, L^*(u' - u'_n),$$
$$v^* = 13\, L^*(u' - u'_n),$$

with:

$$u' = u,$$
$$v' = 1.5\, v,$$

where:

$$u = \frac{4x}{x + 15y + 3z},$$
$$v = \frac{6y}{x + 15y + 3z}.$$

5. Color-difference formulae

Over the time many color difference formulae have been proposed and here the most important, today still in use, are recalled.

5.1 Euclidean color-difference formulae defined on CIE 1976 uniform color spaces

In factories, the measurement of the color difference between a color reference and a color imitation is a daily task, that regards psychometric colorimetry. One of the aims of the CIELAB and CIELUV systems was to give an algorithm for the color difference computation. The CIE formulae given in 1976 are Euclidean, supposing that these spaces have uniform color scales. Furthermore, these Euclidean formulae are the same for CIE 1931 and CIE 1964 observers. The visual situation for these formulae is defined by CIE as follows (Supplement 2 to CIE Publication 15 (1978)):

"Euclidean distances in CIELAB (or CIELUV) color space can be used to represent approximately the perceived magnitude of color differences between object color stimuli of the same size and shape, viewed in identical white to middle-grey surroundings, by an observer photopically adapted to a field of chromaticity not too different from that of average daylight. In cases of deviating conditions, the correlation between calculated and perceived color differences may be impaired."

The CIE 1976 L*u*v* color difference, denoted by ΔE^*_{uv}, is defined as the Euclidean distance between the points representing them in the L*u*v* space and calculated as equation:

$$\Delta E^*_{uv} = \sqrt[2]{(\Delta L^*)^2 + (\Delta u^*)^2 + (\Delta v^*)^2},$$

with:

$$\Delta L^* = L^*_s - L^*_b$$
$$\Delta u^* = u^*_s - u^*_b$$
$$\Delta v^* = u^*_s - u^*_b$$

where the suffix "s" denotes the standard sample (reference) and "b" the test (batch).

The CIE 1976 L*a*b* color difference, denoted by ΔE^*_{ab}, is defined as the Euclidean distance between the points representing them in the L*a*b* space and calculated as equation:

$$\Delta E^*_{uv} = \sqrt[2]{(\Delta L^*)^2 + (\Delta a^*)^2 + (\Delta b^*)^2},$$

with:

$$\Delta L^* = L^*_s - L^*_b,$$
$$\Delta a^* = a^*_s - a^*_b,$$

$$\Delta b^* = b_s^* - b_b^*,$$

In cylindrical coordinates the differences $\Delta L^*, \Delta C_{ab}^*$ and ΔH^*_{ab} are defined, and the color difference formula becomes:

$$\Delta E_{ab}^* = \sqrt{(\Delta L^*)^2 + (\Delta C_{ab}^*)^2 + (\Delta H_{ab}^*)^2},$$

where the chroma difference is:

$$\Delta C_{ab}^* = C_{ab,s}^* - C_{ab,b}^*,$$

and the absolute hue-difference is:

$$|\Delta H_{ab}^*| = \sqrt{(\Delta E_{ab}^*)^2 - (\Delta L^*)^2 - (\Delta C_{ab}^*)^2},$$

with ΔE_{ab}^*defined by equation above. For small color differences the difference of hue $|\Delta H_{ab}^*|$can be expressed by the difference in the hue angles:

$$|\Delta H_{ab}^*| = \sqrt{C_{ab,s}^* C_{ab,b}^*}|\Delta h_{ab}|\left(\frac{\pi}{180}\right) \text{ with } |\Delta h_{ab}| = |h_{ab,s} - h_{ab,b}| \text{ [deg]}.$$

5.2 Non-Euclidean color difference formulae defined on CIELAB

The Euclidean color difference formulae defined in CIELAB and CIELUV did not satisfy the necessities of the factories, thus over time many other formulae have been proposed for small color differences. These formulae are defined on the CIELAB space and are the same for CIE 1931 and CIE 1964 observers. All these formulae are dependent on some parametric factors, that have to be defined on the experimental condition. For the CIEDE2000 formula, the reference conditions for these parametric factors to be set equal to 1 are the following (Luo et al. (2001)):

- Illumination: source simulating the spectral relative irradiance of CIE standard illuminant D65.
- Illuminance: 1000 lx.
- Observer: normal color vision.
- Background field: uniform, neutral grey with L* = 50.
- Viewing mode: object.
- Sample size: greater than 4 degrees subtended visual angle.
- Sample separation: minimum sample separation achieved by placing the sample pair in direct edge contact.
- Sample color-difference magnitude: 0 to 5 CIELAB units.

- Sample structure: homogeneous color without visually apparent pattern or non-uniformity.

We can consider these conditions good for all the following non-Euclidean formulae.

The Colour Measurement Committee (CMC) of the Society of Dyers and Colourists (UK) recommended a color difference formula that has been integrated into some ISO standards (Clarke et al. (1984)). The CMC formula is conceived for the CIELAB system and is mainly based on color difference data known as BFD [Bradford] (ISO 105-J03:1995 (1995). This formula is obtained by introducing different weightings for $\Delta L^*, \Delta C^*_{ab}$ and ΔH^*_{ab} (defined above), that destroy the original Euclidean nature. They are defined as follows:

$$\Delta E_{cmc}(l:c) = \sqrt{\left(\frac{\Delta L^*}{lS_L}\right)^2 + \left(\frac{\Delta C^*{}_{ab}}{cS_C}\right)^2 + \left(\frac{\Delta H^*{}_{ab}}{S_H}\right)^2},$$

where:

$$S_L = \begin{cases} 0.040975 \dfrac{L_1^*}{1 + 0.01765 L_1^*} & \text{for } L_1^* \geq 16 \\ 0.511 \text{ for } L_1^* < 16 \end{cases},$$

$$S_C = \frac{0.0638\, C^*_{\mathrm{ab},1}}{1 + 0.0131 C^*_{\mathrm{ab},1}} + 0.638,$$

$$S_H = S_C(Tf + 1 - f),$$

with:

$$f = \sqrt{\frac{\left(C^*_{\mathrm{ab},1}\right)^4}{\left(C^*_{\mathrm{ab},1}\right)^4 + 1900}},$$

$$T = \begin{cases} 0.36 + \left|0.4\cos(h_{\mathrm{ab},1} + 35)\right| \text{for } h_{\mathrm{ab},1} \geq 345^\circ \text{ or } h_{\mathrm{ab},1} \leq 164^\circ \\ 0.56 + \left|0.2\cos(h_{\mathrm{ab},1} + 168)\right| \text{ for } 164 < h_{\mathrm{ab},1} < 345^\circ \end{cases}.$$

The parametric factors are mostly chosen $c = 1$ and l varied between 1 and 2. The choice of l and c must be indicated by setting the right numbers in the name of the formula, e.g., for textiles a choice of CMC(2:1) is in common use. This formula is now an ISO standard. The CMC formula was a forerunner for the successive ones.

The development of the CIE94 color difference formula was made considering the CMC one as a model (Luo and Rigg (1986)). The CIE 94 formula introduced weighting factors to the lightness, chroma and hue differences, $\Delta L^*, \Delta C^*_{ab}$ and ΔH^*_{ab} of the CIELAB Euclidean formula. The main deviations of the CIE94 formula from the CMC one are in the weighting factors that are much more simple mathematically and do not contain hue-dependent correction terms. The resulting recommendation is as follows:

$$\Delta E *_{94} (k_L : k_C : k_H) = \sqrt{\left(\frac{\Delta L^*}{k_L S_L}\right)^2 + \left(\frac{\Delta C^*_{\text{ab}}}{k_C S_C}\right)^2 + \left(\frac{\Delta H^*_{\text{ab}}}{k_H S_H}\right)^2},$$

where the weighting functions, S_L, S_C, S_H adjust the internal non-uniform structure of the CIELAB formula using:

$$S_L = 1,$$
$$S_C = 1 + 0.045\, C^*_{ab,s},$$
$$S_H = 1 + 0.015\, C^*_{ab,s}.$$

If the standard and the batch of a sample pair is not clearly defined, C^*ab may be replaced by the geometric mean $\left(C^*_{ab,s}\, C^*_{ab,b}\right)^{1/2}$.
The parametric factors, k_L, k_C, k_H are correction terms for variation in experimental conditions. Under reference conditions they are all set to 1.
Today, CIE considers the CIE 1994 color difference formula obsolete because it has been superseded by the CIEDE2000 formula.

The CIEDE2000 total color difference formula corrects for the non-uniformity of the CIELAB color space for small color differences under the above defined reference conditions (Luo et al. (2001)). The CIELAB space was considered inadequate to represent small color differences and a new space is derived from CIELAB, spanned by new coordinates $L'a'b', C', h'$:

$$L' = L^*,$$

$$a' = (1 + G)a^* \text{with } G = 0.5\left(1 - \sqrt{\frac{\bar{C}_{\mathrm{ab}}^{*\ 7}}{\bar{C}_{\mathrm{ab}}^{*\ 7} + 25^7}}\right),$$

$$b' = b^*,$$

$$C' = \sqrt{a'^2 + b'^2},$$

$$h' = tan^{-1}\left(\frac{b'}{a'}\right) \text{ [deg].}$$

Improvements to the calculation of total color difference for industrial color difference evaluation are made through corrections on perceived color difference for the effects of lightness dependence, chroma dependence, hue dependence and hue-chroma interaction, weighted by a factor R_T. The CIEDE2000 color difference formula is defined as follows:

$$\Delta E_{00} = \sqrt{\left(\frac{\Delta L'}{k_L S_L}\right)^2 + \left(\frac{\Delta C'}{k_C S_C}\right)^2 + \left(\frac{\Delta H'}{k_H S_H}\right)^2 + R_T\left(\frac{\Delta C'}{k_C S_C}\right) \rightleftarrows \left(\frac{\Delta H'}{k_H S_H}\right)}$$

where the weighting functions, S_L, S_C and S_H adjust the total color-difference for variation in perceived magnitude with variation in the location of the color-difference pair in L', a', b' coordinates:

$$S_L = 1 + \frac{0.015 \rightleftarrows (\rightleftarrows \bar{L}' - 50)^2}{\sqrt{20 + (\rightleftarrows \bar{L}' - 50)^2}},$$

$$S_C = 1 + 0.045 \rightleftarrows \bar{C}',$$

$$S_H = 1 + 0.015 \rightleftarrows \bar{C}'T$$

with:

$$T = 1 - 0.17\,cos(\bar{h}' - 30°) + 0.24\,cos(2\bar{h}') + 0.32\,cos(3\bar{h}' + 6°) - 0.20\,cos(4\bar{h}' - 63°),$$

and:

$$R_C = 2\sqrt{\frac{\bar{C}'^7}{\bar{C}'^7 + 25^7}},$$

$$R_T = -R_C \, sin(2\Delta\theta) \text{ with } \Delta\theta = 30 \, exp\left[-\left(\frac{\bar{h}' - 275°}{25}\right)^2\right],$$
$$\Delta L' = L'_b - L'_s,$$
$$\Delta C' = C'_b - C'_s,$$
$$\Delta H' = 2\sqrt{C'_b C'_s} \, sin\left(\frac{\Delta h'}{2}\right) \text{with } \Delta h' = h'_b - h'_s.$$

As is written in the above reference conditions, the parametric factors k_L, k_C and k_H are correction terms for variation in experimental conditions and under reference conditions they are all set to 1.
CIE recommends the use the CIEDE2000 formula whenever in the past the CIE 94 or CMC formula were used and such a recommendation is in agreement with the persons who developed the CMC formula.

6. Numerical calculations in C++

For scientists keen in programming, we have developed and tested the following programs in C++ providing the numerical calculations for:

1. TestTristimulusfromIlluminantSpectra: Tristimulus values of illuminant A and D65 starting from Spectral Power Distribution as described in Section 3.1.
2. TestXYZtoFundamental: Conversion of a tristimulus value into fundamental systems as described in Chapter 2.
3. TestXYZtoCIE: Conversion of a tristimulus value into CIE ’76 color systems as described in Chapter 4.
4. TestColorDifferenceFormulae: CIE ’76, CMC, CIE ’94 and CIEDE2000 color-difference formulae described in Chapter 5.

Source codes available at: http://mips.di.unimi.it/download.html.

7. Conflict of interest declaration

The authors declare no conflict of interest.

8. Funding source declaration

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10. Short biography of the authors

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Mauro Fiorentini. Mauro Fiorentini graduated in Physical Sciences at the University of Milan in 1979. Since then he worked on software development, mainly in basic software tools, like compilers and interpreters, mathematical software and embedded systems. In 1986 he wrote a book on the C programming language and in the '90s he worked on some Esprit project, being also the director of OMI/CORE. His interests include mathematics, chess and bridge. Today he is working as R&D director at STE industries.

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